BERNOULLI AUTOMORPHISMS
OF FINITELY GENERATED FREE MV-ALGEBRAS

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ABSTRACT. MV-algebras can be viewed either as the Lindenbaum algebras of Łukasiewicz infinite-valued logic, or as unit intervals $[0, u]$ of lattice-ordered abelian groups in which a strong order unit $u > 0$ has been fixed. They form an equational class, and the free n-generated free MV-algebra is representable as an algebra of piecewise-linear continuous functions with integer coefficients over the unit n-dimensional cube. In this paper we show that the automorphism group of such a free algebra contains elements having strongly chaotic behaviour, is the sense that their duals are measure-theoretically isomorphic to a Bernoulli shift. This fact is noteworthy from the viewpoint of algebraic logic, since it gives a distinguished status to Lebesgue measure as an averaging measure on the space of valuations. As an ergodic theory fact, it provides explicit examples of volume-preserving homeomorphisms of the unit cube which are piecewise-linear with integer coefficients, preserve the denominators of rational points, and enjoy the Bernoulli property.

1. PRELIMINARIES

An *MV-algebra* is an algebra $(A, \oplus, \neg, 0)$ such that $(A, \oplus, 0)$ is a commutative monoid and the identities $\neg\neg f = f$, $f \oplus \neg 0 = \neg 0$, and $\neg(\neg f \oplus g) \oplus g = \neg(\neg g \oplus f) \oplus f$ are satisfied. MV-algebras can be viewed either as the Lindenbaum algebras of Łukasiewicz infinite-valued logic [4], [5], or as unit intervals $[0, u]$ of lattice-ordered abelian groups in which a strong order unit $u > 0$ has been fixed [11]. We assume some familiarity with the basics of the theory; the book [7] provides an extensive treatment, but [13] or [11, §2-4] cover material sufficient for our needs.

We are concerned with automorphisms of the free MV-algebra over n generators, which we denote by Free_n. Such an algebra has a representation as a space of piecewise-linear continuous functions with integer coefficients, as follows. A *rational cellular complex over the n-cube* is a finite set W of cells (i.e., compact convex polyhedrons) whose union is $[0, 1]^n$ and such that:

(1) every vertex of every cell of W has rational coordinates;
(2) if $C \in W$ and D is a face of C, then $D \in W$;
(3) every two cells intersect in a common face.

Equip the real unit interval $[0, 1]$ with the operations $a \oplus b = \min(1, a + b)$, $\neg a = 1 - a$, and the constant 0. Then $[0, 1]$ is an MV-algebra, and so is the power $[0, 1]^{[0,1]^n}$ under componentwise operations. By mapping the ith free generator to the ith projection function $x_i : [0, 1]^n \to [0, 1]$, we obtain a representation $\rho : \text{Free}_n \to [0, 1]^{[0,1]^n}$, which is faithful by Chang's completeness theorem [5], [14]. McNaughton's theorem [10], [12] states then that the range of ρ is exactly the set

of those continuous functions $f : [0,1]^n \to [0,1]$ for which there exists a complex as above and affine linear functions $F_j(\bar{x}) = a_j^1 x_1 + \cdots + a_j^n x_n + a_j^{n+1} \in \mathbb{Z}[x_1, \ldots, x_n]$, in 1-1 correspondence with the n-dimensional cells C_j of the complex, such that $f \upharpoonright C_j = F_j$ for each j. In the following we will tacitly identify Free_n with its image under ρ, except when discussing states in §2.

Let σ be an endomorphism of Free_n. The *dual* of σ is the mapping $S : [0,1]^n \to [0,1]^n$ defined by $S(p)_i = (\sigma(x_i))(p)$. This is equivalent to saying that there exists a complex W as above and matrices $P_j \in \mathrm{Mat}_{n+1}(\mathbb{Z})$, in 1-1 correspondence with the n-dimensional cells C_j of W, such that:

(1) every P_j has last row $(0 \ldots 0\,1)$;
(2) P_j expresses $S \upharpoonright C_j$ in homogeneous coordinates (i.e., if $p = (r_1, \ldots, r_n) \in C_j$ and $S(p) = (s_1, \ldots, s_n)$, then $(s_1 \ldots s_n\,1)^{tr} = P_j(r_1 \ldots r_n\,1)^{tr}$).

As in the proof of [13, Theorem 3.4], σ is an automorphism iff S is injective and all P_j's are in $\mathrm{GL}_{n+1}(\mathbb{Z})$. In particular, all duals of automorphisms preserve Lebesgue measure λ on $[0,1]^n$; we call these duals *McNaughton homeomorphisms* of the unit cube. We say that an automorphism is *Bernoulli* if the corresponding McNaughton homeomorphism is isomorphic to a two-sided Bernoulli shift. This means that there exists a probability space (X, μ) and a measure-theoretic isomorphism $\varphi : ([0,1]^n, \lambda) \to (X^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ such that $\varphi \circ S = B \circ \varphi$, where B is the shift on $X^{\mathbb{Z}}$.

2. Statement of the result

The main result of this paper is the construction of an example (actually, a family of examples) of Bernoulli automorphisms of Free_2. This is the content of Theorem 2.1, whose statement and proof use some amount of ergodic theory, for which we refer to [8] or [18]. In this section we explain the significance of our result both from the viewpoint of ergodic theory and from that of algebraic logic.

Theorem 2.1. *There is an explicitly constructible family $\{\beta_{lm} : 1 \le l, m \in \mathbb{N}\}$ of automorphisms of Free_2 such that, for every β_{lm} in the family, the dual McNaughton homeomorphism B_{lm} has the following properties:*

(1) B_{lm} *is ergodic with respect to Lebesgue measure on the 2-cube;*
(2) B_{lm} *is isomorphic to a Bernoulli shift;*
(3) B_{lm} *fixes pointwise the boundary of $[0,1]^2$ and permutes the set of points having rational coordinates and given denominator;*
(4) B_{lm} *is chaotic in the sense of Devaney (i.e., is topologically transitive, has sensitive dependence on initial conditions, and the set of points having finite orbit is dense in $[0,1]^2$).*

It is a classic result of Oxtoby and Ulam that in the space of measure-preserving homeomorphisms of the unit cube with the uniform topology the subset of ergodic ones is G_δ comeager. The same property holds for the subset of chaotic homeomorphisms; see [1] for a presentation of these results. In particular, we are assured of the existence of many (from the topological point of view) such homeomorphisms. Note however that these theorems are obtained by Baire category methods, and do not provide us with any explicit example. To the best of our knowledge, in the literature one finds two main examples:

(1) The topological skew products in [21], obtained by conjugating the Besicovitch mapping. They are topologically transitive homeomorphisms that

do not preserve Lebesgue measure and restrict to irrational rotations on the boundary.

(2) Quotients of Anosov diffeomorphisms of the 2-torus give chaotic homeo-morphisms of the sphere, which in turn yield chaotic homeomorphisms of $[0,1]^2$ by blowing up a fixed point. The resulting maps have two fixed points on the boundary (one attracting and one repelling, corresponding to the eigenspaces of the original diffeomorphism). This construction is modified in [9], obtaining Bernoulli diffeomorphisms of the unit disk. Another mod-ification is in [3], where the authors start from a linked twist of the torus, and obtain a chaotic homeomorphism of $[0,1]^2$ which fixes the boundary.

Theorem 2.1 allows us to add our family B_{lm} to the above list: it is worth noting that the arithmetical nature of McNaughton homeomorphisms —namely, their piecewise-linearity, and the restriction to integer coefficients— makes them much more rigid objects than the examples (1) and (2), which exploit the full power of the real numbers.

From the viewpoint of algebraic logic, our result gives a distinguished status to Lebesgue measure as an averaging measure on the space $[0,1]^n$ of truth-value assignments. Recall, first of all, the representation ρ described in the previous section. Let $f(x_1, \ldots, x_n)$ be an n-variable formula in Łukasiewicz infinite-valued logic, which we identify with an element of Free_n. We assign to f an "average truth-value" $m(f)$ by setting

$$(A) \qquad m(f) = \int_{[0,1]^n} \rho(f)\, d\lambda,$$

where λ is Lebesgue measure on the n-cube. We have $m(0) = 0$, $m(1) = 1$, and $m(f \oplus g) = m(f) + m(g)$ provided that $\neg(\neg f \oplus \neg g) = 0$: hence the map $m : \mathrm{Free}_n \to [0,1]$ is an example of a *state*, as defined in [13]. By Theorem 2.2 below m does not depend on the choice of a free generating set $\{x_1, \ldots, x_n\}$ for Free_n.

Theorem 2.2. *Let $n \geq 2$. There exists exactly one state on Free_n, namely m, that satisfies the following two conditions:*

(B1) $m(\sigma(f)) = m(f)$, *for every automorphism σ of Free_n;*

(B2) *if, for every $g \in \mathrm{Free}_n$, $f \wedge g = 0$ implies $g = 0$, then*

$$\lim_{k \to \infty} m(\underbrace{f \oplus \cdots \oplus f}_{k \text{ summands}}) = 1.$$

Proof. By [15, Proposition 1.1], for every MV-algebra A the set of states on A is in 1–1 correspondence with the set of regular Borel probability measures on $\mathrm{MaxSpec}\, A$, the latter being the space of maximal ideals of A equipped with the hull-kernel topology. Write $\mathrm{Cont}\, X$ for the MV-algebra of all $[0,1]$-valued continuous functions on the topological space X, under pointwise operations. The dual of the faithful representation $\rho : \mathrm{Free}_n \to \mathrm{Cont}[0,1]^n$ is the map

$$R : [0,1]^n \to \mathrm{MaxSpec}\, \mathrm{Free}_n$$

$$t \mapsto \{f \in \mathrm{Free}_n : \big[\rho(f)\big](t) = 0\},$$

which is a homeomorphism by [11, Lemma 8.1]. It follows that the states on Free_n are in 1–1 correspondence with the Borel probability measures on $[0,1]^n$ (which are necessarily regular); in this bijection, m corresponds to Lebesgue measure λ,

by (A). Hence, our claim boils down to the fact that Lebesgue measure is the only
Borel measure μ on $[0,1]^n$ satisfying the analogues of (B1) and (B2) (in which, of
course, Free_n is replaced by $\rho[\mathrm{Free}_n]$ and m by integration w.r.t. μ). This fact is
proved in [15, Theorem 2.4]. □

The restriction $n \geq 2$ in Theorem 2.2 is irrelevant, since we can always add
a dummy variable to any 1-variable formula; see the discussion about coherence
in [15, §2]. On the other hand, it is crucial to note that, although the proof of
[15, Theorem 2.4] exploits a fair amount of the piecewise-linear structure of $[0,1]^n$,
Theorem 2.2 is an intrinsic, purely algebraic characterization of m. As a matter of
fact, we *define* m as the only state on Free_n that satisfies (B1) and (B2). Note also
that, given any representation of Free_n as an MV-algebra of continuous functions
over a topological space X, Theorem 2.2 uniquely determines a distinguished prob-
ability measure on X, regardless of any arithmetical or piecewise-linear structure
X may or may not have.

Let us further clarify this point: let X be any compact Hausdorff space, and let θ
be any faithful representation of Free_n over X (i.e., θ is an injective homomorphism
of Free_n onto a subalgebra of $\mathrm{Cont}\, X$). Without loss of generality, $\theta[\mathrm{Free}_n]$ separates
the points of X (if not, replace X by an appropriate quotient space). By [7,
Theorem 3.4.3], the map

$$T : X \to \mathrm{MaxSpec}\,\mathrm{Free}_n$$

$$x \mapsto \{f \in \mathrm{Free}_n : \big[\theta(f)\big](x) = 0\}$$

is a continuous bijection. The state m determines then a measure μ on X, namely
the pull-back along T of the measure on $\mathrm{MaxSpec}\,\mathrm{Free}_n$ associated to m. Note
that μ is Borel regular with respect to the topology $\mathcal{T}$ pulled back by T ($\mathcal{T}$ may
be coarser than the original topology on X). In equivalent terms, μ can either be
viewed as the pull-back along $R^{-1} \circ T$ of Lebesgue measure on $[0,1]^n$, or as the
unique $\mathcal{T}$-Borel regular measure on X that satisfies the following:

$$\int_X \theta(f)\,d\mu = \int_{[0,1]^n} \rho(f)\,d\lambda, \qquad \text{for every } f \in \mathrm{Free}_n.$$

As a simple example, let $n = 1$, $X = [0,1]$, and let θ map the free generator x_1
of Free_1 to the function $\big[\theta(x_1)\big](x) = x^3$. We let x and t vary in X and in $[0,1]$,
respectively; of course $\big[\rho(x_1)\big](t) = t$. By standard duality, $R^{-1} \circ T$ must obey

$$\big[\theta(x_1)\big](x) = \big[\rho(x_1)\big]\big((R^{-1} \circ T)(x)\big),$$

and hence $(R^{-1} \circ T)(x) = x^3$. It follows that the μ-measure of, say, the interval
$[\sqrt[3]{t_1}, \sqrt[3]{t_2}] \subseteq X$ is the λ-measure of the interval $[t_1, t_2] \subseteq [0,1]$. In this specific
example $R^{-1} \circ T$ is differentiable, and we can recast things in terms of volume
forms. To compute $m(x_1)$ we integrate $\big[\theta(x_1)\big](x)$ not against the form dx, but
against the image of dt under the cotangent map induced by $R^{-1} \circ T$, namely
$3x^2\,dx$, and we obtain, of course, the value $1/2$.

Having completed our discussion about the state m, we resume our standing
identification of Free_n with $\rho[\mathrm{Free}_n]$. Let q be any density on $[0,1]^n$ (i.e., any
measurable function from the unit cube to the nonnegative real numbers whose
integral w.r.t. λ is 1). Then q induces a measure μ_q on $[0,1]^n$ by $d\mu_q = q\,d\lambda$,
and the function m_q assigning to f the value of the integral of $\rho(f)$ w.r.t. μ_q is
again a state. By possibly adding a dummy variable, we may assume that n is

even; since the Bernoulli property is preserved under direct products [8, Ch. 10 §1], Theorem 2.1 immediately guarantees the existence of a Bernoulli automorphisms σ of Free_n. The dual S of σ is mixing [18, Corollary 4.33.1], and hence the push forward $S_*^k \mu_q$ of μ_q by S^k converges to λ in the weak* topology [18, §4.9 and Theorem 6.12(ii)]. In particular we get

$$\lim_{k\to\infty} m_q(\sigma^k(f)) = \lim_{k\to\infty} \int \rho(\sigma^k(f))\, d\mu_q = \lim_{k\to\infty} \int \rho(f) \circ S^k \, d\mu_q$$
$$= \lim_{k\to\infty} \int \rho(f) \, dS_*^k \mu_q = \int \rho(f) \, d\lambda = m(f).$$

Hence the action of our family makes m an attractor for the family of all states induced by a density: every "wrong" average truth-value assignment eventually converges to the right one.

3. THE FAMILY B_{lm}

Consider the following two complexes over $[0,1]^2$:
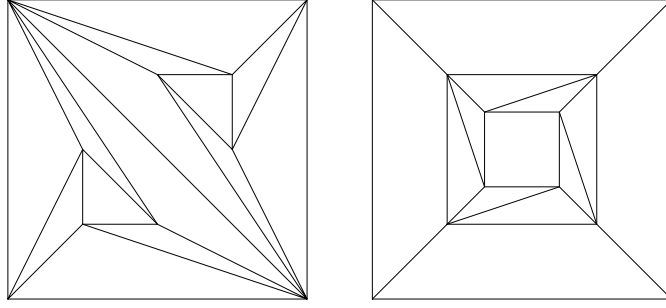
In the first complex, the vertices of the lower inner triangle are $p_0 = (1/4, 1/4)$, $p_1 = (1/2, 1/4)$, $p_2 = (1/4, 1/2)$, and the picture is symmetric under a π rotation about $(1/2, 1/2)$. For $0 \le i \le 2$, let p_i' be the vertex symmetric to p_i. Then there exists a unique homeomorphism R_1 such that:

(1) p_i is mapped to $p_{i+1 \pmod 3}$, and analogously for p_i';
(2) every other vertex is fixed;
(3) R_1 is affine linear on each cell.

By direct computation one sees that R_1 is a McNaughton homeomorphism.

In the second complex, the innermost square has vertices $q_0 = (3/8, 3/8)$, $q_1 = (5/8, 3/8)$, $q_2 = (5/8, 5/8)$, $q_3 = (3/8, 5/8)$, while the intermediate one has vertices p_0, $p_3 = (3/4, 1/4)$, p_0', $p_4 = (1/4, 3/4)$ (see the picture of the intermediate square before the proof of Lemma 5.2). Define R_2 to be the unique homeomorphism that maps q_i to $q_{i-1 \pmod 4}$, fixes every other vertex, and is affine linear on each cell. R_2 is not a McNaughton homeomorphism since, e.g., the matrix that maps the triangle $\langle p_0, q_1, q_0 \rangle$ to $\langle p_0, q_0, q_3 \rangle$ has rational non-integer entries.

Lemma 3.1. R_2^2 is a McNaughton homeomorphism.

Proof. Let B, C be 2-dimensional cells in the second complex, with corresponding matrices P, Q, respectively. It suffices to show that if the topological interior of $B \cap R_2^{-1}C$ is not empty, then QP has integer entries. Observe that the

complex is symmetric under a clockwise $\pi/2$ rotation about $(1/2, 1/2)$, and this rotation is expressible in homogeneous coordinates via a matrix $A \in \mathrm{GL}_3(\mathbb{Z})$. We can clearly restrict our attention to the triangles in the ring bounded by the innermost and the intermediate squares. We then see easily that, fixing $B = \langle p_0, p_3, q_1 \rangle$ and $C = \langle p_0, q_1, q_0 \rangle$, we only have to check that the four matrices $P^2, QP, (APA^{-1})Q, (AQA^{-1})Q$ have integer entries, and this can be done by hand computation. $\square$

For every $1 \leq l, m$, let $B_{lm} = R_2^{4m} \circ R_1^{3l}$, and let β_{lm} be the associated automorphism of Free$_2$. The family of all β_{lm}'s is the one in the statement of Theorem 2.1.

4. PROOF OF THEOREM 2.1

We fix $l, m \geq 1$, and let $\beta = \beta_{lm}$, $B = B_{lm}$. We first observe that β is explicitly constructible. Indeed, by using the theory of Schauder hats [12, p. 598], [14, §3], we can write down algorithmically syntactic expressions for the formulas $t_1 = x_1 \circ R_1$, $t_2 = x_2 \circ R_1$, $t_3 = x_1 \circ R_2^2$, $t_4 = x_2 \circ R_2^2$. The images $\beta(x_1)$ and $\beta(x_2)$ of the free generators x_1, x_2 of Free$_2$ are then (huge) formulas obtained from $t_1, \ldots, t_4$ via recursive nesting.

Since R_1 and R_2 fix the boundary of $[0, 1]^2$, so does B. Define the *denominator* of the rational point $p = (r_1, r_2) \in [0, 1]^2 \cap \mathbb{Q}^2$ to be the unique integer $d \geq 1$ such that dr_1, dr_2, d are relatively prime integers. From the definitions, it is immediate that every McNaughton homeomorphism permutes the set A_d of all points having denominator d. Since every A_d is finite, and the union of all A_d's is the set of rational points, which is dense in the square, it follows that every topologically transitive McNaughton homeomorphism is chaotic in the sense of Devaney [1, §4.5]; the statements (3) and (4) in Theorem 2.1 are then consequences of (1) and (2). Note that, since McNaughton homeomorphisms are determined by their actions on the A_d's, the automorphism group of Free$_2$ is residually finite, and hence acts chaotically on some Hausdorff space [3, Theorem 1]. Theorem 2.1(4) says that in this group there are specific elements that act chaotically, and that the space on which the action takes place may be assumed to be the unit square.

To prove the statements (1) and (2) in Theorem 2.1, we use Pesin theory [16]: the key ingredients will be Lemma 4.1 and Lemma 4.2, stated below.

Lemma 4.1. *There is a B-invariant and eventually strictly invariant cone field U_q, defined λ-almost everywhere on $[0, 1]^2$, and depending measurably on q.*

By [20, p. 152], [6, Theorem III.4.4], Lemma 4.1 entails that for λ-all points $q \in [0, 1]^2$ the tangent space at q decomposes in the direct sum $E_q^s \oplus E_q^u$ of stable and unstable subspaces, the Lyapunov exponents are defined at q, and their sum is 0. These facts determine foliations of the square into local stable and unstable curves γ_q^s and γ_q^u. Due to the piecewise linearity of B, these (un)stable leafs are indeed segments, as in [19, Proposition 1], and the resulting foliations are absolutely continuous [19, p. 71]. Using [16, Theorems 7.2, 7.9, 8.1] and the argument in [17], Theorem 2.1 will result from the following lemma.

Lemma 4.2. *For λ-all $p, q \in [0, 1]^2$, there exist $h_0, k_0 \geq 0$ such that, for every $h \geq h_0$ and $k \geq k_0$, we have*

$$B^{-h}[\gamma_p^s] \cap B^k[\gamma_q^u] \neq \emptyset.$$

We prove the above lemmas in the next two sections.

5. Proof of Lemma 4.1

Let D be the unit disk in polar coordinates (r, θ). Let $f : [0, 1] \to [0, 1]$ be the continuous piecewise-fractional function defined by

$$f(r) = \min\big(1, (1/r - 1)/3\big),$$

and let $F : D \to D$ be the twist mapping $F(r, \theta) = (r, \theta + 2\pi l f(r))$. Let $L \subset [0, 1]^2$ be the triangle with vertices $(0, 0), (1, 0), (0, 1)$, and let $M : D \to L$ be the homeomorphism defined by

$$M(0, \theta) = (1/3, 1/3);$$

$$M(1, \theta) = \begin{cases} (1 - 3\theta/2\pi, 3\theta/2\pi), & \text{if } 0 \le \theta < 2\pi/3; \\ (0, 2 - 3\theta/2\pi), & \text{if } 2\pi/3 \le \theta < 4\pi/3; \\ (3\theta/2\pi - 2, 0), & \text{if } 4\pi/3 \le \theta < 2\pi; \end{cases}$$

$$M(r, \theta) = (1 - r)M(0, \theta) + rM(1, \theta).$$

Lemma 5.1. *M conjugates F with $R_1^{3l} \restriction L$ (i.e., $M \circ F = R_1^{3l} \circ M$).*

Proof. This is a straightforward computation, which we just sketch. We fix without loss of generality $1/4 < r < 1$. By definition of R_1, the points of the triangle $T(r) = \{M(r, \theta) : 0 \le \theta < 2\pi\}$ "rotate counterclockwise" along $T(r)$ under the action of R_1. We have to check that the speed of the rotation is $f(r)/3$, upon normalizing the lengths of the sides of $T(r)$ to be $r/3$. Since R_1 is piecewise affine, we may do the checking only for the 6 points in the intersection of $T(r)$ with the 1-dimensional simplices in the complex defining R_1. Consider, e.g., the point w in $T(r) \cap \langle (0, 0), p_2 \rangle$. It moves to z in $T(r) \cap \langle (0, 0), p_0 \rangle$ with speed $\overline{wz}/(\text{length of } T(r)) = \overline{wz}/r$. Since $\overline{wz}/(1 - r) = \overline{p_2 p_0}/(3/4) = (4^{-1}/3)/(3/4) = 1/9$ (because the triangles $\langle (0, 0), w, z \rangle$ and $\langle (0, 0), p_2, p_0 \rangle$ are similar, and $p_0, p_2 \in T(4^{-1})$), we obtain $\overline{wz}/r = (1 - r)/(9r) = f(r)/3$, as desired. $\qquad \square$

If A is a π rotation of the unit square about $(1/2, 1/2)$, $L' = AL$, and $M' = A \circ M$, then clearly M' conjugates F with $R_1^{3l} \restriction L'$.

Define now $g : [0, 1] \to [0, 1]$ and $G : D \to D$ by

$$g(r) = \min(1, 1/r - 1),$$
$$G(r, \theta) = (r, \theta - 2\pi m g(r)).$$

Let $E \subset [0, 1]^2$ be the square $\langle p_0, p_3, p_0', p_4 \rangle$, and let $N : D \to E$ be the homeomorphism

$$N(r, \theta) = (1/2, 1/2) + \begin{cases} r(1/4, -1/4 + \theta/\pi), & \text{if } 0 \le \theta < \pi/2; \\ r(3/4 - \theta/\pi, 1/4), & \text{if } \pi/2 \le \theta < \pi; \\ r(-1/4, 5/4 - \theta/\pi), & \text{if } \pi \le \theta < 3\pi/2; \\ r(-7/4 + \theta/\pi, -1/4), & \text{if } 3\pi/2 \le \theta < 2\pi. \end{cases}$$

The mapping N arises from the same construction as the mapping M: we have just given directly its analytic expression, as the reader can easily verify. The same proof as in Lemma 5.1 shows that N conjugates G with $R_2^4 \restriction E$.

Consider the annuli $H = \{(r, \theta) : 1/4 < r < 1\}$, $K = \{(r, \theta) : 1/2 < r < 1\}$ and the standard row vector basis $(\partial/\partial r|_{(r,\theta)} \; \partial/\partial \theta|_{(r,\theta)})$ on their tangent spaces. The

differentials $T_{(r,\theta)}F$ and $T_{(r,\theta)}G$ have then the matrix form

$$T_{(r,\theta)}F = \begin{pmatrix} 1 & 0 \\ -2\pi l/3r^2 & 1 \end{pmatrix}, \qquad T_{(r,\theta)}G = \begin{pmatrix} 1 & 0 \\ 2\pi m/r^2 & 1 \end{pmatrix}.$$

We introduce the matrices

$$A_1(r) = \begin{pmatrix} 1 & 0 \\ \pi l/3r^2 & -1 \end{pmatrix}, \qquad A_2(r) = \begin{pmatrix} 1 & 0 \\ -\pi l/3r^2 & -1 \end{pmatrix},$$

$$A_3(r) = \begin{pmatrix} 1 & 0 \\ -\pi m/r^2 & 1 \end{pmatrix}, \qquad A_4(r) = \begin{pmatrix} 1 & 0 \\ \pi m/r^2 & 1 \end{pmatrix}.$$

For each $i = 1,2$ (respectively, $i = 3,4$), let $U^i_{(r,\theta)}$ be the cone in $T_{(r,\theta)}H$ (respectively, in $T_{(r,\theta)}K$) of all vectors whose coordinates in basis $(\partial/\partial r|_{(r,\theta)} \quad \partial/\partial\theta|_{(r,\theta)})A_i(r)$ are both ≥ 0 or both ≤ 0. Then $T_{(r,\theta)}F$ maps bijectively $U^1_{(r,\theta)}$ onto $U^2_{F(r,\theta)}$, and $T_{(r,\theta)}G$ maps bijectively $U^3_{(r,\theta)}$ onto $U^4_{G(r,\theta)}$. In particular, since $U^2_{(r,\theta)} \subset U^1_{(r,\theta)}$ and $U^4_{(r,\theta)} \subset U^3_{(r,\theta)}$, the differential $T_{(r,\theta)}F$ maps $U^2_{(r,\theta)}$ into $U^2_{F(r,\theta)}$, and $T_{(r,\theta)}G$ maps $U^4_{(r,\theta)}$ into $U^4_{G(r,\theta)}$. Recall that a cone is *strictly contained* in another cone if the former is contained —except for the common vertex— in the topological interior of the latter. Consider the parallelograms $P_1, \ldots, P_4$ in the following picture:
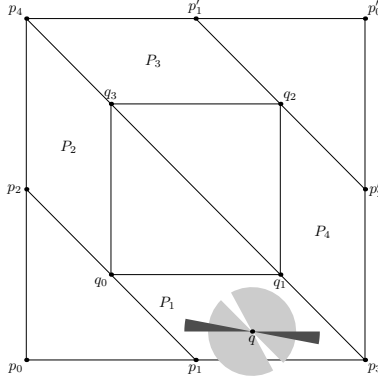
Lemma 5.2. *Let $q = (x, y)$ be in the topological interior of $P_1 \cup P_2$. Then the cone $(T_{M^{-1}q}M)[U^2_{M^{-1}q}]$ is strictly contained in $(T_{N^{-1}q}N)[U^3_{N^{-1}q}]$, and $(T_{N^{-1}q}N)[U^4_{N^{-1}q}]$ is strictly contained in $(T_{M^{-1}q}M)[U^1_{M^{-1}q}]$.*

Proof. Four computations are needed, according whether q is in P_1 or P_2: it takes about one hour to perform all four using a computer algebra system. We will carry on just one computation, the others being analogous. Namely, we will show that $(T_{N^{-1}q}N)[U^4_{N^{-1}q}]$ is strictly contained in $(T_{M^{-1}q}M)[U^1_{M^{-1}q}]$, on the assumption that $q = M(r_1, \theta_1) = N(r_2, \theta_2)$ is in P_1; hence $0 < \theta_1 < 2\pi/3$ and $3\pi/2 < \theta_2 < 2\pi$ (see the picture above, in which the cones are computed for $l = m = 1$). Introducing the basis $(\partial/\partial x|_{(x,y)} \quad \partial/\partial y|_{(x,y)})$ on $T[0,1]^2$, the differentials $T_{(r_1,\theta_1)}M$ and $T_{(r_2,\theta_2)}N$ are given by the matrices

$$B_1 = \begin{pmatrix} 2/3 - 3\theta_1/2\pi & -3r_1/2\pi \\ -1/3 + 3\theta_1/2\pi & 3r_1/2\pi \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -7/4 + \theta_2/\pi & r_2/\pi \\ -1/4 & 0 \end{pmatrix}.$$

We then have to check that the images of the columns of $A_4(r_2)$ under B_2 have coordinates all > 0 or all < 0 in the basis given by the images of the columns of $A_1(r_1)$ under B_1: in other words, that the matrix

$$\left[B_1 A_1(r_1) \right]^{-1} \left[B_2 A_4(r_2) \right]$$

has all entries > 0 or all < 0. Observe that

$$A_1(r_1)^{-1} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/r_1^2 & -1 \end{pmatrix}$$

for a certain $a > 0$ (a depends on r_1 and l), and analogously

$$A_4(r_2) = \begin{pmatrix} 1 & 0 \\ 3/r_2^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

for some $b > 0$, depending on r_2 and m. Hence, it is sufficient to check the entries of

$$\begin{pmatrix} 1 & 0 \\ 1/r_1^2 & -1 \end{pmatrix} B_1^{-1} B_2 \begin{pmatrix} 1 & 0 \\ 3/r_2^2 & 1 \end{pmatrix}.$$

We can multiply our matrices by factors > 0 to get rid of denominators: in particular, we compute

$$(*) \qquad 3r_1^2 \begin{pmatrix} 1 & 0 \\ 1/r_1^2 & -1 \end{pmatrix} B_1^{-1} = \begin{pmatrix} 9r_1^2 & 9r_1^2 \\ 9r_1\theta_1 - 2r_1\pi + 9 & 9r_1\theta_1 - 4r_1\pi + 9 \end{pmatrix}.$$

From the definition of M we have the identity

$$\begin{pmatrix} x - 1/3 \\ y - 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 & -3/2\pi \\ -1/3 & 3/2\pi \end{pmatrix} \begin{pmatrix} r_1 \\ r_1\theta_1 \end{pmatrix},$$

which we invert and substitute in the right-hand side of $(*)$, obtaining

$$C = \begin{pmatrix} \begin{matrix} 81x^2 + 81y^2 + 162xy- \\ -108x - 108y + 36 \end{matrix} & \begin{matrix} 81x^2 + 81y^2 + 162xy- \\ -108x - 108y + 36 \end{matrix} \\ 6\pi y - 2\pi + 9 & -6\pi x + 2\pi + 9 \end{pmatrix}.$$

Analogously, we compute

$$2\pi r_2 B_2 \begin{pmatrix} 1 & 0 \\ 3/r_2^2 & 1 \end{pmatrix}$$

and apply to the result the substitution $r_2 = -4y + 2$, $r_2\theta_2 = \pi x - 7\pi y + 3\pi$ that follows from inverting the identity

$$\begin{pmatrix} x - 1/2 \\ y - 1/2 \end{pmatrix} = \begin{pmatrix} -7/4 & 1/\pi \\ -1/4 & 0 \end{pmatrix} \begin{pmatrix} r_2 \\ r_2\theta_2 \end{pmatrix}.$$

We call D the resulting matrix, namely

$$D = \begin{pmatrix} 2\pi x - \pi + 6 & 32y^2 - 32y + 8 \\ 2\pi y - \pi & 0 \end{pmatrix}.$$

We finally compute CD and apply to the result the change of variables $x = z - w + 1/2$, $y = w + 1/4$, so that $q = (z, w)$, with $0 < z < 1/4$ and $0 < w < 1/8$. We thus get the matrix

$$\begin{pmatrix} (9/32)(12z+1)^2(4\pi z - \pi + 12) & (9/8)(4w-1)^2(12z+1)^2 \\ \begin{matrix} (1/2)\left[(4\pi^2 + 36\pi)z + (-8\pi^2 + 72\pi)w+ \right. \\ \left. +\pi^2 - 15\pi + 108 \right] \end{matrix} & (4w-1)^2(12\pi w - \pi + 18) \end{pmatrix}$$

which has all entries > 0 in the above range of values for z and w.   □

Of course the analogue of Lemma 5.2 (with M' replacing M) holds for $q \in P_3 \cup P_4$. We define a cone field U_q on $T[0,1]^2$ as follows:

$$U_q = \begin{cases} (T_{M^{-1}q}M)[U^2_{M^{-1}q}], & \text{if } q \in MH \setminus (\text{closure of } NK); \\ (T_{M'^{-1}q}M')[U^2_{M'^{-1}q}], & \text{if } q \in M'H \setminus (\text{closure of } NK); \\ (T_{N^{-1}q}N)[U^4_{N^{-1}q}], & \text{if } q \in NK. \end{cases}$$

U_q is defined λ-almost everywhere and depends piecewise-continuously on q. Let Σ be the set of all q's such that the cone field is defined and B is differentiable on all the bilateral orbit of q: clearly Σ has full measure. For $q \in \Sigma$ there are four possibilities:

  (a) $q, Bq \in (MH \cup M'H) \setminus NK$;
  (b) $q \in (MH \cup M'H) \setminus NK, Bq \in NK$;
  (c) $q \in NK, Bq \in (MH \cup M'H) \setminus NK$;
  (d) $q, Bq \in NK$.

Using Lemma 5.2 one checks immediately that in all cases $(T_q B)[U_q]$ is contained in U_{Bq}, i.e., the cone field is invariant under the differential $T_q B$. Moreover, in all cases except (a) the containment is strict. Almost every point $(r, \theta) \in H$ has an F-orbit which is dense in the circle of radius r. This implies that almost every $q \in \Sigma$ enters in NK under R_1^3, and hence one of the cases (b), (c), (d) must occur along the forward B-orbit of q. The cone field U_q is therefore eventually strictly invariant, and this concludes the proof of Lemma 4.1.

6. Proof of Lemma 4.2

We use a geometric argument, analogous to those in [2, §5] and [19, Lemma 6] for the toral case. Here the situation is more delicate, since we do not have the universal covering of the torus at our disposal. Nevertheless, we may use the homeomorphism $(r, \theta) \mapsto N(2r, \theta)$ from D to $[0,1]^2$ to obtain a ramified covering of $[0,1]^2$ by $[0,1] \times \mathbb{R}$ in the obvious way: keeping in mind this covering makes the argument below easier to follow.

The forward iterates of local unstable leaves are broken segments, and are all similar to each other (because all of them tend to approximate the unstable foliation). Analogously for the backwards iterates of stable leaves. The following pictures of such approximations (unstable and stable, respectively) are obtained for $l = m = 1$, but clearly the pattern does not change for higher values of l and m.
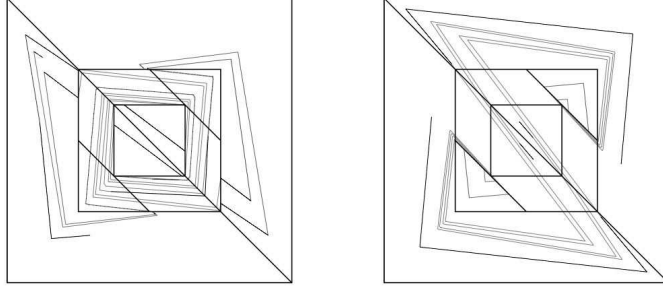
Let P_i be one of the four parallelograms in the preceding section. The *horizontal sides* of P_i are those fixed by R_2, and the *vertical sides* those fixed by R_1. A broken segment η is *horizontal in P_i* if some component of $\eta \cap P_i$ connects the vertical sides of P_i; we have a dual definition of η being *vertical in P_i*. A simple geometric reasoning shows that if η is horizontal in some P_i, then $B^k[\eta]$ is horizontal in all P_i's, for every $k \geq 2$ (if at least one of l, m is > 1, then $k \geq 1$ is sufficient). Analogously, if η is vertical in some P_i, then $B^{-h}[\eta]$ is vertical in all P_i's for every $h \geq 2$.

Since horizontal segments intersect vertical ones, it is sufficient to show that $B^k[\gamma^u]$ is horizontal in P_i, for some i and $k \geq 0$, and analogously $B^{-h}[\gamma^s]$ is vertical in some P_i; here $\gamma^u = \gamma^u_q$ and $\gamma^s = \gamma^s_p$ for $p, q \in \Sigma$ as in the preceding section. We treat only the unstable case, the stable one being similar. The length of $B^k[\gamma^u]$ goes to infinity as k increases. Since the twists R_1 and R_2 "go in the same direction" (as in Case 1 of [19, Theorem 1]), $B^k[\gamma^u]$ will not remain contained in any small region (the dual phenomenon, in which the twists "go in opposite directions" and $B^k[\gamma^u]$ remains contained, is described in [17, p. 346]). In particular, for k sufficiently large, $B^k[\gamma^u]$ will cross some parallelogram, say P_3. The geometry of the unstable cone field forces $B^k[\gamma^u]$ to cross P_3 either horizontally, or through $\langle q_3, q_2 \rangle$ and $\langle p_4, q_3 \rangle$, or through $\langle p_4, p'_1 \rangle$ and $\langle p'_1, q_2 \rangle$. In the first case we are done, so suppose there is a component η_0 of $B^k[\gamma^u] \cap P_3$ that connects $\langle q_3, q_2 \rangle$ with $\langle p_4, q_3 \rangle$. Let a_0 be the extreme of η_0 that is in $\langle p_4, q_3 \rangle$, and b_0 the extreme that is in $\langle q_3, q_2 \rangle$. Compute $\eta_1 = R_1^{3l} \eta_0$, $\eta_2 = R_2^{4m} \eta_1 = B\eta_0$, $\eta_3 = R_1^{3l} \eta_2$, ..., and define:

- if j is even, $a_{j+1} = a_j$, $b_{j+1} =$ the first point, moving from a_{j+1} towards $R_1^{3l} b_j$ along η_{j+1}, in which η_{j+1} crosses $\langle q_3, q_2 \rangle$;
- if j is odd, $b_{j+1} = b_j$, $a_{j+1} =$ the first point, moving from b_{j+1} towards $R_2^{4m} a_j$ along η_{j+1}, in which η_{j+1} crosses $\langle p_4, q_3 \rangle$.

The points a_j and b_j divide η_j in three parts, a central one which remains in P_3 and approaches q_3, and two lateral ones, which get longer as j increases. In particular the "left part" (the one that has a_j as an extreme) will eventually cross P_2. Fix now j to be the first index for which this happens, and let c_j be the first point, moving from a_j along the left part, in which η_j intersect a side of P_2. Again by the geometry of the unstable cone field, either $c_j \in \langle p_2, q_0 \rangle$ and j is even, or $c_j \in \langle p_4, p_2 \rangle$ and j is odd. If $c_j \in \langle p_2, q_0 \rangle$, then η_j is horizontal in P_2 and we are done. If $c_j \in \langle p_4, p_2 \rangle$, then the image under R_2^{4m} of the part of η_j between c_j and b_j is horizontal in P_3, and again we are done.

We left open the possibility that there is a component η_0 of $B^k[\gamma^u] \cap P_3$ that connects $\langle p_4, p'_1 \rangle$ with $\langle p'_1, q_2 \rangle$, but the proof is analogous. This time we take a_0 (respectively, b_0) to be the extreme of η_0 in $\langle p'_1, q_2 \rangle$ (respectively, $\langle p_4, p'_1 \rangle$), and

construct η_j, a_j, b_j as in the preceding case. The part of η_j that has a_j as an extreme will eventually cross the triangle $\langle p_0', p_1', p_2' \rangle$ and the parallelogram P_4. Again, η_j must enter P_4 through $\langle q_2, p_2' \rangle$ and leave P_4 either at $c_j \in \langle q_1, p_3 \rangle$ with j even, or at $c_j \in \langle q_2, q_1 \rangle$ with j odd. But then either η_j (in the first case) or the image under R_2^{4m} of the part of η_j between b_j and c_j (in the second case) crosses P_4 horizontally. This concludes the proof of Lemma 4.2 and hence of Theorem 2.1.

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